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First-order resonance in the reflection of baroclinic Rossby waves

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The nonlinear interaction between an incident and a reflected Rossby wave produces a steady flow parallel to the (non-zonal) reflecting wall and a transient flow oscillating at twice the frequency of the incident–reflected pair. If the transient forcing is resonant, i.e. a free Rossby wave, the resonant response must have zero amplitude at the wall in order to fulfil the boundary condition there; a straightforward expansion predicts a linear growth of its amplitude in the offshore direction y. Resonance is possible only if $0 < |\sin\alpha| \leqslant \frac{1}{3}$, where α is the angle between the wall and the easterly direction. This requirement is met by several boundaries in the ocean. A simple graphical method to find a resonant triad is described.

Using the method of multiple scales, it is shown that the wave amplitudes of the triad are slowly varying periodic functions of y, such that the energy flux of the triad through any plane parallel to the wall vanishes, as required by energy conservation. The waves participating in the resonant triad become wave packets. The three waves do not exchange energy in time due to the additional constraint on the motion imposed by the boundary condition at the wall. It is shown that the wave amplitudes cannot be slowly varying functions of y and time.

As a possible oceanic application of the theoretical findings, the distance from the wall where one would expect to find large semi-annual amplitudes if annual Rossby waves are impinging on the boundary is of the order of 100 km. Motivated by similar studies (Plumb 1977; Mysak 1978), there are speculations on what would happen if three incident–reflected Rossby wave pairs (or modes) are taken, allowing each mode amplitude to be slowly varying in time.

1. Introduction

The weak nonlinear interaction between an incident and a reflected baroclinic Rossby wave leads to (Graef-Ziehl 1990 hereinafter referred to as GZ): (i) an Eulerian steady flow parallel to the (non-zonal) reflecting wall and (ii) a transient flow oscillating at twice the frequency (2ω) of the Rossby wave pair. The main interest in GZ was to study the Eulerian mean flow that is driven by the waves (a first-order effect), the influence that the steady flow had on the driving Rossby waves themselves (a second-order effect) and the steady flow occurring at third order driven in part by the modified Rossby waves. It was shown in GZ that the transient forcing could be resonant if $0 < |\sin\alpha| \leqslant \frac{1}{3}$, where α is the angle between the reflecting wall and the easterly direction. The goal of this paper is to answer the following question: what happens if the nonlinear interaction of an incident and a reflected baroclinic Rossby wave excites a free Rossby wave? Specifically: (i) Given the possibility of having the incident, reflected and forced waves forming a resonant triad, how to find such waves? (ii) Is there a particular solution that grows linearly in time being consistent with the constraints on the motion? (iii) It is possible to find a uniformly valid solution in the resonant case?

In the weak nonlinear interaction regime, there is great interest in studying resonant interactions, for all non-resonant interactions only produce a small-amplitude background noise of forced waves whose amplitudes are small compared to those waves produced by the resonant interactions (Pedlosky 1987).

The theory of resonant interactions among Rossby waves has a long history. Longuet-Higgins & Gill (1967) studied resonant interactions of barotropic, divergent (free surface) Rossby waves in a laterally unbounded ocean. If the Rossby waves are baroclinic, account must be taken of the nonlinear coupling between vertical normal modes. As regards boundary effects, Plumb (1977) considered resonant interactions of barotropic Rossby waves in a *zonal* channel to study their stability. Mysak (1978) studied resonant interactions of topographic Rossby waves in a continuously stratified channel of arbitrary orientation; however, his scaling is such that the leading order balance is the linear quasi-geostrophic potential vorticity equation (QGPVE) without the β-term.

To the author's knowledge, resonant interactions of *baroclinic* planetary waves in an ocean with a single lateral boundary have not been studied.

An important difference between this work and the papers of Plumb (1977) and Mysak (1978) is that both authors took a triad of *wave modes* (solutions to the linear problem), whereas here the resonant triad is an incident, a reflected and the 2ω Rossby wave. For instance, in Plumb (1977) a wave mode is in fact a superposition of two Rossby waves (an incident–reflected pair) with the cross-channel wavenumbers discretized. Of course, one reason why Plumb considered more than one wave mode is that one wave mode is an exact nonlinear solution (this is not true for a non-zonal wall). In both papers each wave mode amplitude was slowly varying in time, thereby implying an energy exchange among the members of the resonant triad. It will be shown that the incident, reflected and 2ω Rossby waves do not exchange energy due to the constraint on the motion imposed by the boundary condition at the wall, even though the resonant conditions are satisfied.

The plan of this paper is as follows. In §2 a brief background to the theory that motivated this study is presented. A general discussion about the resonantly interacting triad is given in §3, i.e. describing a graphical method to find a resonant triad, together with a particular solution that grows linearly with distance from the wall. In §4 the technique of multiple scales is used to obtain a uniform perturbation expansion. The physical meaning behind the wave amplitude equations of the resonant triad and the integral constraints derived from them is shown via energy considerations in §5. Finally, §6 is devoted to discussion and conclusions.

2. Theoretical background

The coordinate system has x parallel and y perpendicular to the wall, and z vertically upwards. The governing equation is the QGPVE, and the boundary conditions are no normal flow at the boundaries (reflecting wall, flat bottom and rigid lid) and the solution must be bounded at infinity. A perturbative solution for the (non-dimensional) quasi-geostrophic streamfunction is sought in the form

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \dots, \tag{2.1}$$

where $\epsilon = U/(\beta L^2)$ is the β-Rossby number in which β is the northward gradient of the planetary vorticity and L and U are horizontal length and horizontal velocity scales, respectively.

If the leading-order solution is the superposition of an incident and a reflected Rossby wave, i.e. if

$$\psi^{(0)} = A\Psi_n(z)(\cos\theta_1^{(0)} - \cos\theta_2^{(0)}), \tag{2.2}$$

where A is the wave amplitude, $\Psi_n(z)$ is an eigenfunction of the vertical Sturm–Liouville problem with corresponding eigenvalue λ_n, $\theta_i^{(0)} \equiv kx + l_i y - \omega t + \phi$, $i = 1, 2$ and

$$\omega = \sigma_n(k, l_i) \equiv \frac{-(k\cos\alpha + l_i \sin\alpha)}{k^2 + l_i^2 + L^2 f_0^2 \lambda_n}, \quad i = 1, 2, \tag{2.3}$$

then the first-order $[O(\epsilon)]$ perturbation equations are

$$\partial_t \left\{ \nabla^2 \psi^{(1)} + \partial_z \left[\frac{1}{S(z)} \partial_z \psi^{(1)} \right] \right\} + \cos\alpha\, \partial_x \psi^{(1)} + \sin\alpha\, \partial_y \psi^{(1)}$$

$$= -B_{12}\Psi_n^2(z)[\cos(\theta_1^{(0)} - \theta_2^{(0)}) - \cos(\theta_1^{(0)} + \theta_2^{(0)})], \tag{2.4}$$

where $S(z) = H^2 N^2(z)/f_0^2 L^2$ is the stratification parameter (Pedlosky 1987), f_0 the Coriolis parameter, H the water depth, $N(z)$ the Brunt–Väisälä frequency, $B_{12} = \frac{1}{2}A^2 k(l_1 - l_2)(l_1^2 - l_2^2)$,

$$\partial_x \psi^{(1)} = 0 \quad \text{at} \quad y = 0, \tag{2.5}$$

$$\partial_t \partial_z \psi^{(1)} = -J(\psi^{(0)}, \partial_z \psi^{(0)}) \equiv 0 \quad \text{at} \quad z = -1, 0 \tag{2.6}$$

and

$$\psi^{(1)} \quad \text{bounded as} \quad x \to \pm\infty, y \to \infty. \tag{2.7}$$

Henceforth it will be assumed that $B_{12} \neq 0$. For $A \neq 0$ this is equivalent to assuming that $k \neq 0$ and $|l_1| \neq |l_2|$ (i.e. $\sin\alpha \neq 0$ or non-zonal walls).

The solution for $\psi^{(1)}$ is written as $\psi^{(1)} = \psi^{(1)}_{\text{hom}} + \psi^{(1)}_{p1} + \psi^{(1)}_{p2}$, where

$$\psi^{(1)}_{p1} = -\frac{B_{12}\Psi_n^2(z)}{(l_1 - l_2)\sin\alpha} \sin[(l_1 - l_2)y] \tag{2.8}$$

is the steady forced solution that gives an Eulerian steady flow parallel to the (non-zonal) wall, first described in Mysak & Magaard (1983) for their special case of no friction,

$$\psi^{(1)}_{p2} = \sum_{m=0}^{\infty} \frac{b_m}{\lambda - L^2 f_0^2 \lambda_m} \Psi_m(z) \sin(\theta_1^{(0)} + \theta_2^{(0)}) \equiv F(z)\sin(\theta_1^{(0)} + \theta_2^{(0)}), \tag{2.9}$$

in which

$$b_m = -B_{12} \int_{-1}^{0} \Psi_n^2 \Psi_m \, \mathrm{d}z/(2\omega) \equiv -B_{12}\xi_{nnm}/(2\omega)$$

and

$$\lambda \equiv -\{(2k)^2 + (l_1 + l_2)^2 + (1/2\omega)[2k\cos\alpha + (l_1 + l_2)\sin\alpha]\}, \tag{2.10}$$

and $\psi^{(1)}_{\text{hom}}$ is a homogeneous solution of (2.4), which is chosen to satisfy (2.5).

Solution (2.9) is valid when λ is not one of the eigenvalues $L^2 f_0^2 \lambda_m$, which is a sufficient condition for not having resonance.

In order to have resonance, it is necessary that:

(i) λ be an eigenvalue, say $\lambda = L^2 f_0^2 \lambda_M$, because then, and only then, $2\omega = \sigma_M(2k, l_1 + l_2)$, and

(ii) the projection of the forcing function $\Psi_n^2(z)$ on the Mth mode eigenfunction be non-zero, i.e. $\xi_{nnM} \neq 0$.

Furthermore, it is necessary that $|\sin\alpha| \leqslant \frac{1}{3}$ to have a solution of the resonance conditions with k real (GZ; shown below).

The issue of this paper is to study what happens when there is resonance, i.e. when all the conditions above are met and an acceptable solution to the resonance conditions is found.

3. Solution of the resonance conditions

To satisfy the resonance conditions there are three equations, namely, $\omega = \sigma_n(k, l_i)$, $i = 1, 2$ and $2\omega = \sigma_M(2k, l_1 + l_2)$, and six unknowns: k, l_1, l_2, ω, n and M.

The dispersion relations $\omega = \sigma_n(k, l_i)$, $i = 1, 2$ imply $l_1 + l_2 = -\sin\alpha/\omega$; if this is substituted into $2\omega = \sigma_M(2k, l_1 + l_2)$ one arrives at an equation $\mathscr{F}(k, \omega, M) = 0$. Solving for k, one gets

$$k_{\pm}^{(\mathrm{res})} = -\frac{\cos\alpha}{8\omega} \pm \frac{1}{2}\left(\frac{1 - 9\sin^2\alpha}{(4\omega)^2} - L^2 f_0^2 \lambda_M\right)^{\frac{1}{2}}. \tag{3.1}$$

The condition $|\sin\alpha| \leqslant \frac{1}{3}$ is obvious. Thus, if the resonance conditions are satisfied then $k = k_{\pm}^{(\mathrm{res})}$; however, the converse is not always true: for that, $k_{\pm}^{(\mathrm{res})}$ should also be in the interval (k_2, k_1), where

$$k_{1,2} = -\frac{\cos\alpha}{2\omega} \pm \left(\frac{1}{4\omega^2} - L^2 f_0^2 \lambda_n\right)^{\frac{1}{2}}, \tag{3.2}$$

to assure that $l_{1,2}$ are real and different. If $k = k_{\pm}^{(\mathrm{res})}$ (either) and $k_{\pm}^{(\mathrm{res})} \in (k_2, k_1)$ then, and only then, are the resonance conditions satisfied.

In the theory of the North Hawaiian Ridge Current by Mysak & Magaard (1983), the authors used $\alpha = 25°$ as an average value for the Hawaiian Ridge; this theory would then suggest that resonant interactions (at first order) would be absent owing to the orientation of the ridge.

Note that (3.1) is independent of the mode number n. It would appear as though the number of degrees of freedom were two. However, if a single equation in terms of k and l_1 (of fourth degree in k and l_1) is obtained, λ_n and λ_M appear in it.

Since the (non-dimensional) wave period T of the incident–reflected pair must satisfy $T > T_{c,n} = 4\pi L f_0 \lambda_n^{\frac{1}{2}}$ to have $l_{1,2}$ real, but also

$$T \geqslant \frac{8\pi L f_0 \lambda_M^{\frac{1}{2}}}{(1 - 9\sin^2\alpha)^{\frac{1}{2}}} = \frac{2T_{c,M}}{(1 - 9\sin^2\alpha)^{\frac{1}{2}}} = T_{\min,M} \tag{3.3}$$

in order for $k_{\pm}^{(\mathrm{res})}$ to be real, it follows that $T \geqslant \max(T_{c,n}, T_{\min,M})$. Figure 1 shows $T_{\min,M}$ as a function of $|\alpha|$ for $M = 0$ and for the first five baroclinic modes. The cutoff periods were computed using the eigenvalues obtained from the stratification at weather station November (140° W, 30° N) (Emery & Magaard 1976).

Therefore, for given ω and M (and α such that $|\sin\alpha| \leqslant \frac{1}{3}$), there are at most two triads of Rossby waves that can interact resonantly, the triad being formed by the incoming, reflected and forced wave. Figure 2 shows the wavelengths of the triad as a function of the period T for the barotropic and the first mode baroclinic ($n = M = 1$) case. For annual period incident waves there is a reasonable range of 100 to 900 km in the wavelengths of the triad. Barotropic Rossby waves of periods about 1 month, as suggested by recent altimeter data, would have wavelengths of order 1000 km.

To understand why there should be any constraint on the orientation of the wall to have a resonantly interacting triad, consider the following. Assume that for the given

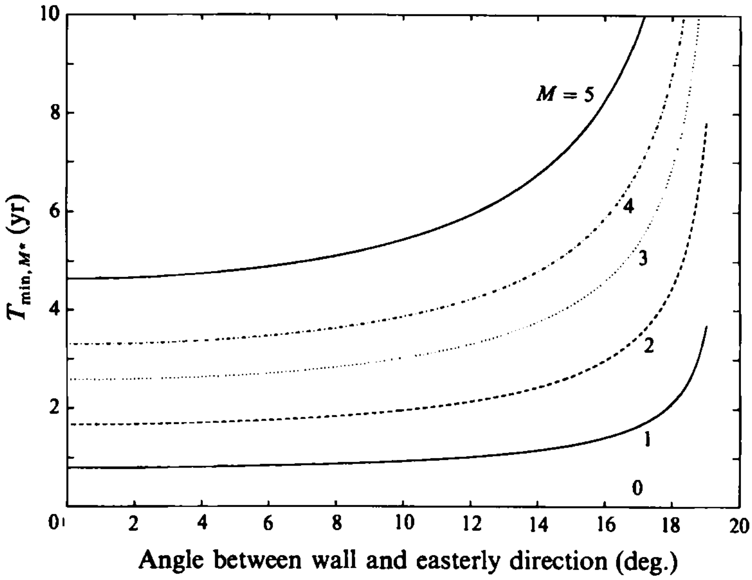

FIGURE 1. The minimum period $T_{\min,M^*} = T_{\min,M}/(\beta L)$ (in years) versus $|\alpha|$ (in degrees) to have resonance. Each curve is labelled according to the mode number M of the 'forced' wave. (Reference latitude $\theta_0 = 25°$.)

M and ω the $(M, 2\omega)$ circle can be drawn, i.e. $A_M^2 = 1/(16\omega^2) - L^2 f_0^2 \lambda_M > 0$. The y-component (η) of all the wave vectors lying on this circle must satisfy

$$|\eta + \sin\alpha/(4\omega)| \leqslant A_M.$$

For resonance to occur, the wave vector with offshore component

$$\eta = l_1 + l_2 (= -\sin\alpha/\omega)$$

must be on the $(M, 2\omega)$ circle, therefore $|3\sin\alpha/(4\omega)| \leqslant A_M$, which is precisely the condition to have $k_{\pm}^{(\text{res})}$ real.

The following procedure describes a graphical method (see figure 3) to find a resonantly interacting triad:

(i) Draw the (n, ω) circle where the wavenumbers of the incident–reflected wave pairs must lie.

(ii) Draw the straight line parallel to the k-axis (parallel to the wall) given by the equation $l = -\sin\alpha/\omega$. Note that all possible pairs of incident–reflected Rossby waves of the same frequency (for the given α) share the same $l_1 + l_2$, thus this sum is an invariant from pair to pair.

(iii) Pick a mode number M and compute A_M^2. If $A_M^2 > 0$ then draw the $(M, 2\omega)$ circle of radius A_M and centred at $(-\cos\alpha/4\omega, -\sin\alpha/4\omega)$. If this circle intersects the line $l = -\sin\alpha/\omega$, then the points of intersection (at most two) have coordinates $[2k_{\pm}^{(\text{res})}, l_1 + l_2]$; otherwise, there are no resonant triads. If $A_M^2 \leqslant 0$ then there are no Mth mode Rossby waves of frequency 2ω that can interact resonantly with the nth mode incident–reflected waves of the given frequency ω. Choose a smaller M.

(iv) The points of intersection of the straight lines $k = k_{\pm}^{(\text{res})}$ with the (n, ω) circle are the tips of the incident and reflected wavenumber vectors. It could happen that neither of the two lines $k = k_{\pm}^{(\text{res})}$ intersects the (n, ω) circle; in these cases there are no resonant triads with $l_{1,2}$ real. Only in the barotropic ($n = 0$) rigid lid ($\lambda_0 = 0$) case is it always true that $k_{\pm}^{(\text{res})} \in (k_2, k_1)$.

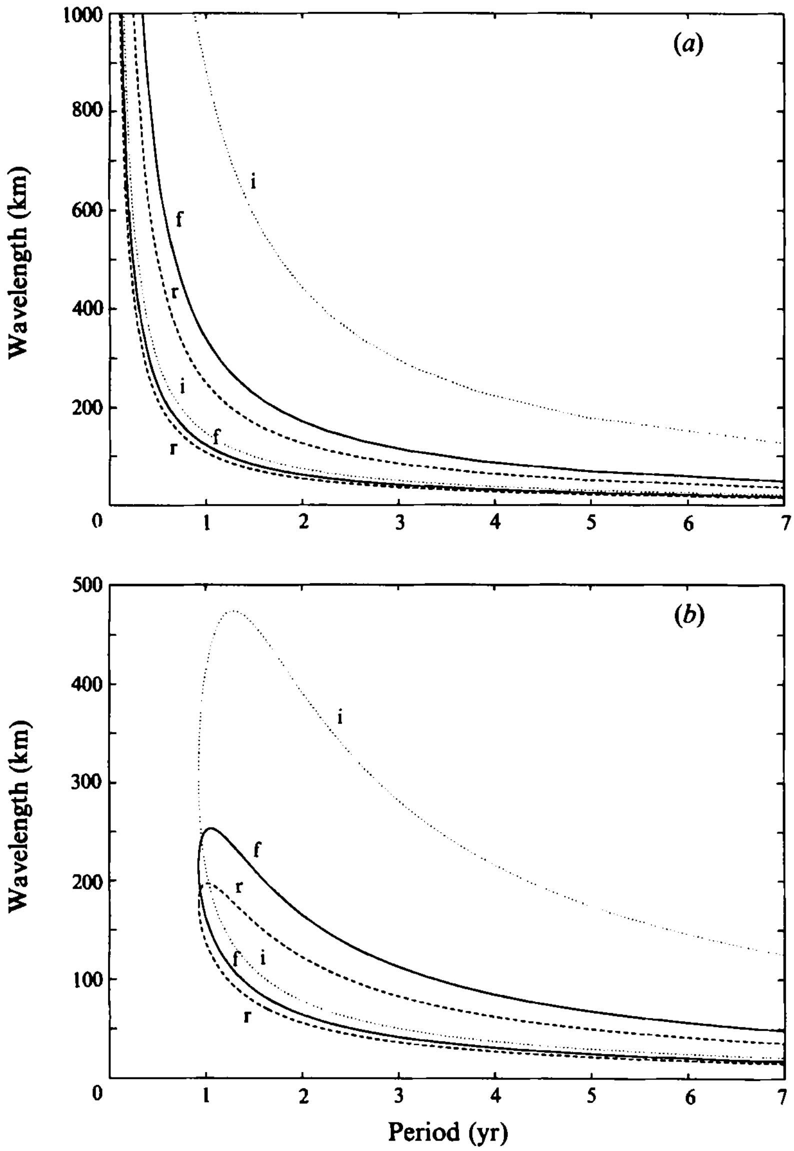

FIGURE 2. The wavelengths (km) of the resonant triad (i = incident, r = reflected and f = forced) versus period (years) of the incident–reflected Rossby wave pair for $\theta_0 = 25°$, $|\alpha| = 10°$. (*a*) $n = 0 \Rightarrow M = 0$; (*b*) $n = 1$ and $M = 1$.

3.1. *A particular solution*

When there is resonant forcing, solution (2.9) needs to be modified for $m = M$. The solution for the Mth mode amplitude when $\lambda = L^2 f_0^2 \lambda_M$ and $\xi_{nnM} \neq 0$ is given by (GZ)

$$\Phi_M^{(1)} = \frac{B_{12}\,\xi_{nnM}}{4\omega(l_1+l_2)+\sin\alpha}\,y\cos(\theta_1^{(0)}+\theta_2^{(0)}). \tag{3.4}$$

The denominator in (3.4) is $4\omega(l_1+l_2)+\sin\alpha = -3\sin\alpha \neq 0$ for non-zonal walls. Solution (3.4) grows linearly away from the boundary. There is a linearly growing solution in y to the zeroth-order problem when the two roots $l_{1,2}$ coalesce (GZ); thus, this is the solution that gets excited when there is resonant forcing.

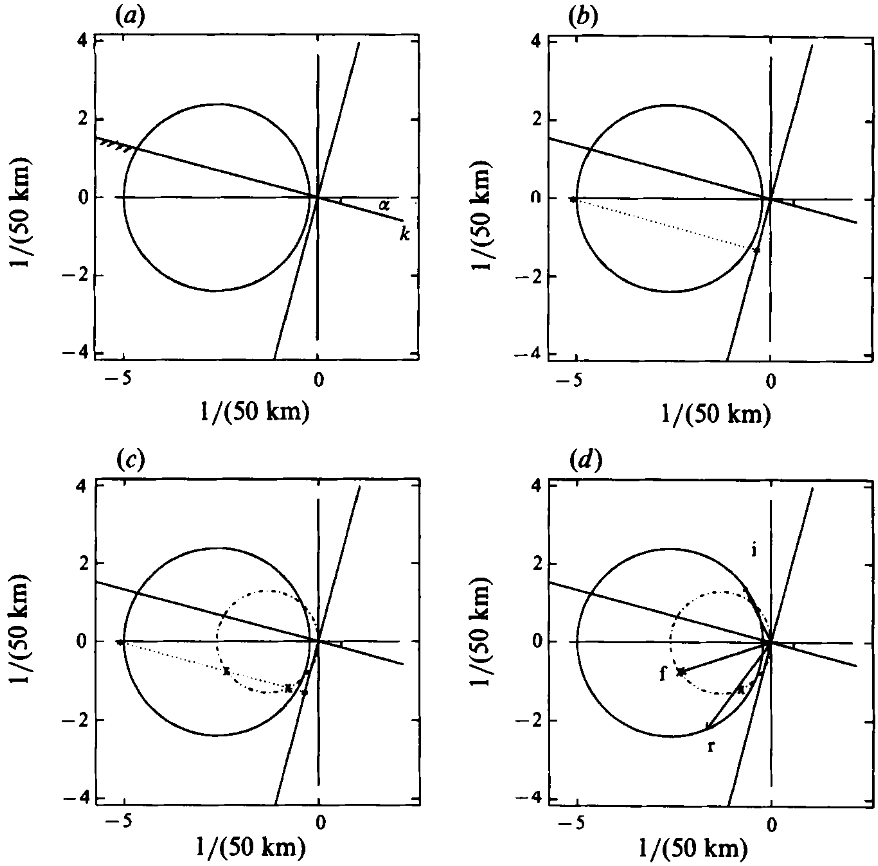

FIGURE 3. Graphical method to find a resonantly interacting triad for given n and ω of incident wave. (a) The (n, ω) circle with $n = 1$, $T = 1$ year; $\alpha = 15°$, $\theta_0 = 25°$. (b) The straight line $l = -\sin\alpha/\omega$ (parallel to wall). (c) The $(M, 2\omega)$ circle, $M = 0$ and its intersections with $l = -\sin\alpha/\omega$ at $(2k_{\pm}^{(\mathrm{res})}, l_1 + l_2)$. ($d$) The wavenumber vectors of a wave triad (i = incident, r = reflected and f = forced).

Therefore, solution (2.9) is replaced by

$$\psi_{p2}^{(1)} = \Phi_M^{(1)}\, \Psi_M(z) + \sum_{\substack{m=0 \\ m \neq M}}^{\infty} \frac{b_m\, \Psi_m(z)}{\lambda - L^2 f_0^2 \lambda_m} \sin(\theta_1^{(0)} + \theta_2^{(0)}). \tag{3.5}$$

The amplitude of oscillation of the particular solution in the case of resonant forcing is unbounded as $y \to \infty$.

There is no solution that grows linearly in time. Indeed, one can show that although

$$\varphi_M^{(1)} = \frac{-B_{12}\, \xi_{nnM}}{(2k)^2 + (l_1 + l_2)^2 + L^2 f_0^2 \lambda_M}\, t \cos(\theta_1^{(0)} + \theta_2^{(0)}) \tag{3.6}$$

is a solution for the Mth mode amplitude, one cannot satisfy the boundary condition at the wall (2.5). This just reflects the fact that the problem under consideration is a boundary value problem periodic in time, inhomogeneous in space (in y) and not an initial value problem. The y-coordinate in this problem plays the role of time in resonant interaction problems that are homogeneous in space (e.g. Pedlosky 1987).

Furthermore, a solution of the form $G(y, t) \cos(\theta_1^{(0)} + \theta_2^{(0)})$ with $G(0, t) = 0$ does not exist, unless G is only a function of y. This means that the amplitudes of the resonant triad cannot be expected to be slow functions of y and t.

The first-order perturbative solution $\psi = \psi^{(0)} + \epsilon\psi^{(1)}$ is clearly not valid for large y. In the next section the method of multiple scales will be applied to obtain a uniformly valid solution in y to $O(\epsilon)$.

Finally, it is interesting to note that the forced wave, $\sim \cos(\theta_1^{(0)} + \theta_2^{(0)})$, is a reflected (incident) wave for a western (eastern) boundary. This result is in agreement with the idea of westward intensification: that a western (eastern) boundary acts as a source (sink) of small-scale motions. In the unbounded case, the energy flux of the forced wave (and of those that produced it) is irrelevant. But here one must be prepared to accept and try to understand physically how one can have, say for a western boundary, two reflected Rossby waves and one incident Rossby wave, as will become clear shortly.

4. Multiple-scale analysis

The method of multiple scales is described in Bender & Orszag (1978) and Nayfeh (1981). The straightforward, pedestrian expansion indicates that very near the coast the amplitude of the forced wave is linear in ϵy and suggests that the amplitudes of the triad be functions of

$$Y_1 = \epsilon y, \tag{4.1}$$

a longer space scale because Y_1 is not negligible when y is of order ϵ^{-1} or larger.

The y-derivatives in the QGPVE are transformed according to $\partial_y \rightarrow \partial_y + \epsilon\,\partial_{Y_1} + \dots$ and $\partial_{yy} \rightarrow \partial_{yy} + 2\epsilon\,\partial_{yY_1} + \dots$. Using these transformations, the QGPVE to $O(\epsilon)$ becomes

$$\partial_t\left\{\nabla^2\psi^{(1)} + \partial_z\left[\frac{1}{S(z)}\partial_z\psi^{(1)}\right]\right\} + \cos\alpha\,\partial_x\psi^{(1)} + \sin\alpha\,\partial_y\psi^{(1)}$$
$$= -J\left\{\psi^{(0)}, \nabla^2\psi^{(0)} + \partial_z\left[\frac{1}{S(z)}\partial_z\psi^{(0)}\right]\right\} - 2\partial_t\partial_{yY_1}\psi^{(0)} - \sin\alpha\,\partial_{Y_1}\psi^{(0)}, \tag{4.2}$$

where $\nabla^2 = \partial_x\partial_x + \partial_y\partial_y$ and $J(A,B) = \partial_x A\,\partial_y B - \partial_y B\,\partial_x A$. The boundary conditions are the same as before for the first-order problem because, except for the Jacobian in the boundary conditions in z, they do not involve y-derivatives.

Following Pedlosky (1987), the leading-order solution is written now as a superposition of three *wave packets* participating in the resonant wave triad, namely

$$\psi^{(0)} = A_1(Y_1,\dots)\,\Psi_n(z)\cos\theta_1^{(0)} - A_2(Y_1,\dots)\,\Psi_n(z)\cos\theta_2^{(0)} + A_3(Y_1,\dots)\,\Psi_M(z)\cos\theta_3^{(0)}, \tag{4.3}$$

where now $\theta_i^{(0)} \equiv kx + l_i y - \omega t + \phi_i(Y_1,\dots)$, $i = 1,2$ and $\theta_3^{(0)} = \theta_1^{(0)} + \theta_2^{(0)}$, $\xi_{nnM} \neq 0$, and the resonant conditions are satisfied, i.e. $\lambda = L^2 f_0^2 \lambda_M$, or, equivalently $2\omega = \sigma_M(2k, l_1 + l_2)$.

The zeroth-order QGPVE does not have Y_1-derivatives so it is satisfied by (4.3); the amplitudes and phases act as constants in the leading-order perturbation equations. However, in order for (4.3) to satisfy the boundary condition at the wall it is necessary to impose conditions on the amplitudes and phases. At $Y_1 = 0$, i.e. at $y = 0$:

$$\left.\begin{aligned} A_1 &= A_2 = A, \\ \phi_1 &= \phi_2 = \phi, \\ A_3 &= 0. \end{aligned}\right\} \tag{4.4}$$

As is usual in multiple-scale analysis, the functional dependence on Y_1 of the amplitudes and phases is unknown at the leading order. It will be determined at $O(\epsilon)$ by eliminating secular terms.

To find a solution to (4.2) $\psi^{(1)}$ is expanded in terms of the complete set of eigenfunctions $\{\Psi_m(z)\}$:

$$\psi^{(1)} = \sum_{m=0}^{\infty} \Phi_m^{(1)}(x, y, t)\, \Psi_m(z), \tag{4.5}$$

where $\Phi_m^{(1)} = \int_{-1}^{0} \psi^{(1)} \Psi_m(z)\, dz$. The equation governing $\Phi_m^{(1)}$ is obtained by multiplying (4.2) by Ψ_m, integrating over the depth and using (2.6); the result is

$$\begin{aligned}
&\partial_t(\nabla^2 - L^2 f_0^2 \lambda_m)\Phi_m^{(1)} + \cos\alpha\, \partial_x \Phi_m^{(1)} + \sin\alpha\, \partial_y \Phi_m^{(1)} \\
&= -\mathscr{B}_{12}\, \xi_{nnm}[\cos(\theta_1^{(0)} - \theta_2^{(0)}) - \cos\theta_3^{(0)}] - \mathscr{B}_{13}\, \xi_{nMm}[\cos\theta_2^{(0)} - \cos(2\theta_1^{(0)} + \theta_2^{(0)})] \\
&\quad - \mathscr{B}_{23}\, \xi_{nMm}[\cos\theta_1^{(0)} - \cos(\theta_1^{(0)} + 2\theta_2^{(0)})] \\
&\quad - (2\omega l_1 + \sin\alpha)\, \delta_{nm}(\partial_{Y_1} A_1 \cos\theta_1^{(0)} - A_1 \partial_{Y_1} \phi_1 \sin\theta_1^{(0)}) \\
&\quad + (2\omega l_2 + \sin\alpha)\, \delta_{nm}(\partial_{Y_1} A_2 \cos\theta_2^{(0)} - A_2 \partial_{Y_1} \phi_2 \sin\theta_2^{(0)}) \\
&\quad - [4\omega(l_1 + l_2) + \sin\alpha]\, \delta_{Mm}[\partial_{Y_1} A_3 \cos\theta_3^{(0)} - A_3 \partial_{Y_1}(\phi_1 + \phi_2) \sin\theta_3^{(0)}],
\end{aligned} \tag{4.6}$$

where

$$\left.\begin{aligned}
\mathscr{B}_{12} &= \tfrac{1}{2} A_1 A_2 k(l_1 - l_2)(l_1^2 - l_2^2), \\
\mathscr{B}_{13} &= -\tfrac{1}{2} A_1 A_3 k(l_2 - l_1)[(2k)^2 + (l_1 + l_2)^2 + L^2 f_0^2 \lambda_M - (k^2 + l_1^2 + L^2 f_0^2 \lambda_n)], \\
\mathscr{B}_{23} &= \tfrac{1}{2} A_2 A_3 k(l_1 - l_2)[(2k)^2 + (l_1 + l_2)^2 + L^2 f_0^2 \lambda_M - (k^2 + l_2^2 + L^2 f_0^2 \lambda_n)].
\end{aligned}\right\} \tag{4.7}$$

Regarding (4.6), the following remarks should be made: (i) The terms proportional to $\sin\theta_i^{(0)}$ and $\cos\theta_i^{(0)}$, $i = 1, 2$ will be secular (i.e. homogeneous solutions of (4.6)) if $m = n$ because they are nth mode Rossby waves. If $m \neq n$ the terms $\sim \delta_{mn}$ disappear but those remaining, $\sim \mathscr{B}_{i3}\, \xi_{nMm}$, $i = 1, 2$, are not secular. (ii) If $m = M$ then, and only then, the terms $\sim \sin\theta_3^{(0)}$, $\sim \cos\theta_3^{(0)}$ are secular, since they are Mth mode Rossby waves. (iii) In summary, the terms with a Kronecker's delta factor survive only when $m = n$ or $m = M$, which is precisely when they are needed to eliminate the secular terms.

From the equation for $m = n$, the removal of secular terms from the right-hand side of (4.6) requires

$$-(2\omega l_1 + \sin\alpha)\, \partial_{Y_1} A_1 - \mathscr{B}_{23}\, \xi_{nMn} = 0, \tag{4.8}$$

$$(2\omega l_2 + \sin\alpha)\, \partial_{Y_1} A_2 - \mathscr{B}_{13}\, \xi_{nMn} = 0, \tag{4.9}$$

$$\partial_{Y_1} \phi_1 = \partial_{Y_1} \phi_2 = 0, \tag{4.10}$$

and from the equation for $m = M$ (which would be the same as that for $m = n$ if $n = M$)

$$-[4\omega(l_1 + l_2) + \sin\alpha]\, \partial_{Y_1} A_3 + \mathscr{B}_{12}\, \xi_{nnM} = 0. \tag{4.11}$$

The remaining equation is $\partial_{Y_1}(\phi_1 + \phi_2) = 0$, which is automatically satisfied by (4.10), whose solution (using (4.4)) is $\phi_1 = \phi_2 = \phi =$ constant as far as the dependence on Y_1 is concerned. Thus, resonant interactions at $O(\epsilon)$ do not change the phase of the waves, but only their amplitudes.

Using the dispersion relations it follows that

$$[(2k)^2 + (l_1 + l_2)^2 + L^2 f_0^2 \lambda_M - (k^2 + l_i^2 + L^2 f_0^2 \lambda_n)] = \frac{\sin\alpha}{2\omega}(-1)^i (l_2 - l_1), \quad i = 1, 2, \tag{4.12}$$

implying that the total wavenumber of the forced wave is always between the total wavenumbers of the incident and reflected waves. The third wave must have intermediate westward slowness in order to have maximum frequency (Ripa 1981). Only in the $n = M$ case is the wavelength of the forced wave always between the wavelengths of the incident and reflected waves (see figure 2).

Noting that $2\omega l_i + \sin\alpha = (-1)^i \omega(l_2 - l_1)$, $i = 1, 2$ and using (4.12), the amplitude equations can be rewritten as

$$\partial_{Y_1} A_1 - \gamma A_2 A_3 = 0, \tag{4.13}$$

$$\partial_{Y_1} A_2 + \gamma A_1 A_3 = 0, \tag{4.14}$$

$$\partial_{Y_1} A_3 - \frac{2\omega(l_1 - l_2)}{3\sin\alpha}\gamma A_1 A_2 = 0, \tag{4.15}$$

where

$$\gamma \equiv \tfrac{1}{2}k\frac{\sin\alpha}{2\omega^2}(l_1 - l_2)\,\xi_{nnM}. \tag{4.16}$$

The system of first-order nonlinear ODEs just obtained is typical of three-wave resonance problems and it has been extensively studied (see Ripa 1981; Craik 1985). Exact solutions of (4.13)–(4.15) are known in terms of elliptic functions. When the three constants multiplying the products $A_i A_j$ have different signs (as here), these solutions are mostly periodic, but there are non-periodic limiting cases. The amplitudes remain bounded whenever the signs of the constants differ.

Generally, the amplitudes depend on time, and in some instances on time and spatial, variables (Newell 1969; Plumb 1977; Craik 1985). However, for this problem, the wave amplitudes cannot be functions only of a slow time variable $T_1 = \epsilon t$ simply because there is no way that the superposition of the incident, reflected and forced wave (but now a free Rossby wave) satisfy the boundary condition at the wall. And, as mentioned in §3.1, neither can the amplitudes be functions of Y_1 and T_1 if $A_3(0, T_1) = 0$ is required.

The group velocity of a Rossby wave $\omega = \sigma_n(\boldsymbol{k})$ can be written as

$$\boldsymbol{c}_g = -2\frac{\omega\boldsymbol{k} + \frac{1}{2}\hat{\boldsymbol{\imath}}_E}{K^2}, \tag{4.17}$$

where $\hat{\boldsymbol{\imath}}_E = \hat{\boldsymbol{\imath}}\cos\alpha + \hat{\boldsymbol{\jmath}}\sin\alpha$ is the unit vector in the eastward direction, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are the unit vectors in the x- and y-directions, respectively, and $K^2 = |\boldsymbol{k}|^2 + L^2 f_0^2 \lambda_n$. In dimensional units $\boldsymbol{c}_{g*} = -2\omega_*[\boldsymbol{k}_* + \beta\hat{\boldsymbol{\imath}}_E/(2\omega_*)]/K_*^2$. It follows that the y-component of $\boldsymbol{c}_g$ for each wave of the triad is $c_{gyi} = (-1)^i \omega(l_1 - l_2)/K_i^2$, $i = 1, 2$ and $c_{gy3} = 3\sin\alpha/K_3^2$. Thus, if l_1 (l_2) is the root of (2.3) with the positive (negative) radical, l_1 (l_2) corresponds to the incident (reflected) wave for all boundary orientations. The expression for c_{gy3} confirms that the wave A_3 is a reflected (incident) wave for a western (eastern) boundary.

There are two functionally independent first integrals of system (4.13)–(4.15), which permits one to reduce its solution to a problem of quadratures and the system is completely integrable (Bessis & Chafee 1986). Multiplying A_1 by (4.13), A_2 by (4.14) and adding the resulting equations yields $\partial_{Y_1}(A_1^2 + A_2^2) = 0$, or $A_1^2 + A_2^2 = 2A^2$, having used the boundary conditions (4.4). Analogously, from (4.14) and (4.15) one gets $2\omega(l_1 - l_2) A_2^2 + 3\sin\alpha A_3^2 = 2\omega(l_1 - l_2) A^2$. Combining these two last equations, which are functionally independent first integrals of (4.13)–(4.15), one obtains

$$c_{gy1} K_1^2 A_1^2 + c_{gy2} K_2^2 A_2^2 + c_{gy3} K_3^2 A_3^2 = 0, \tag{4.18}$$

which means that the sum of the average energy fluxes (or total flux) normal to the wall is a constant equal to zero, as required by energy conservation (shown below in §5).

The numerical solution of the wave amplitudes for different wall orientations and for

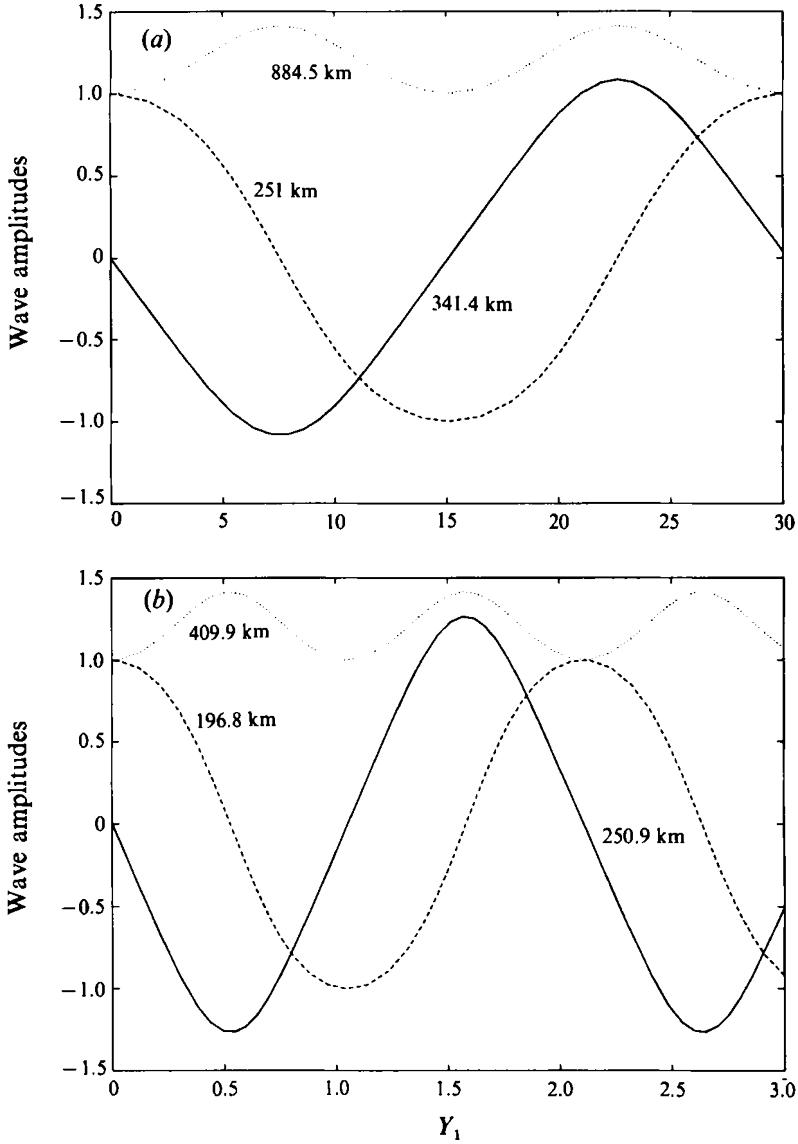

FIGURE 4. The wave amplitudes of a resonant triad as a function of $Y_1 = \epsilon y$. Dotted line: A_1 or incident Rossby wave; dashed line: A_2 or reflected Rossby wave; solid line: A_3 or 'forced' Rossby wave. The corresponding wavelengths are indicated on each curve. Reference latitude $\theta_0 = 25°$. $\alpha = 10°$, $T = 1$ year. (*a*) $n = M = 0$; (*b*) $n = 1$, $M = 1$.

barotropic ($n = 0$) and first baroclinic mode ($n = 1$) annual incident–reflected Rossby wave pairs is shown in figures 4 and 5. For the case shown in figure 4(*b*), where all waves are first-mode baroclinic, the value $\xi_{111} = 1.78$ was used (taken from Flierl 1978); note that $\xi_{111} = 0$ when $N(z)$ is constant. Clearly, the amplitudes or envelopes of the wave packets are periodic in Y_1. The Y_1-axis can be dimensionalized using $y_* = LY_1/\epsilon$.

For western (eastern) boundaries the envelope of the incident (reflected) wave packet A_1 (A_2) is nowhere zero. This can be checked directly from (4.18). The other two envelopes of reflected (incident) wave packets oscillate around zero out of phase and at the same frequency, the amplitude of A_3 always being the greatest of the three. A_3

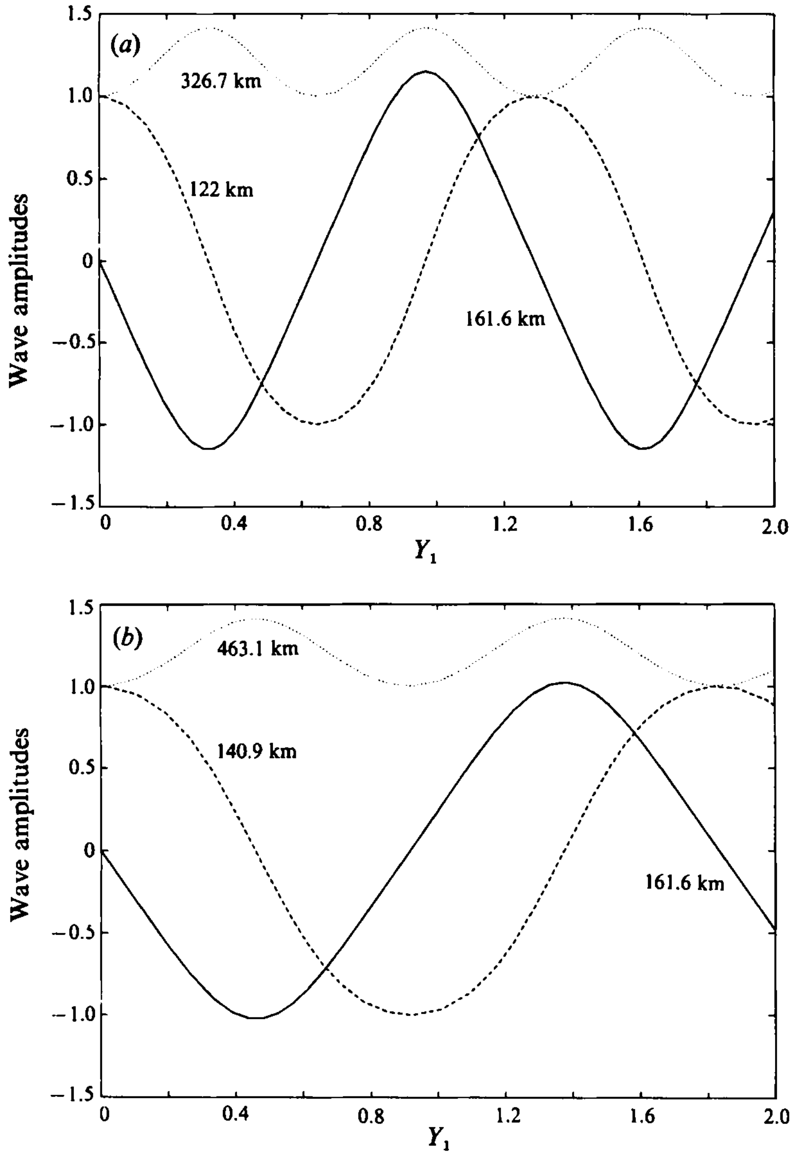

FIGURE 5. As in figure 4, except that $\alpha = 19°$, close to the threshold angle $\sin^{-1}(\frac{1}{3}) = 19.47°$ and (*b*) $n = 1$, $M = 0$.

starts with zero at $Y_1 = 0$ and grows approximately linearly near the coast, as indicated by the pedestrian expansion solution (3.4); it then reaches its maximum at that point where the reflected (incident) wave amplitude is zero.

It would be of interest for oceanographic applications to know at what distance from the boundary one would expect to find large semi-annual period amplitudes if annual Rossby waves are impinging on the boundary. This distance (d_*), taken as the distance from the boundary at which $|A_3|$ attains the first maximum, has been calculated as a function of the maximum horizontal particle speed (at $y = z = 0$) of the incident Rossby wave, U_i, for the four cases shown in figures 4 and 5 (to compute ϵ, a dimensional wave amplitude is needed, which has been calculated from U_i). It may be shown that d_* is inversely proportional to U_i. For $U_i = O(1\ \text{cm/s})$ (see e.g. figure 11 of Kang & Magaard 1980) it is $d_* = O(100 \text{ km})$, except for figure 4(*a*), where the incident barotropic Rossby wave has a wavelength of 884 km and $d_* = O(1000 \text{ km})$.

Recall that the signal produced by the resonant interaction between an incident and a reflected Rossby wave has the structure $A_3(Y_1)\,\Psi_M(z)\cos(\boldsymbol{k}_3\cdot\boldsymbol{x}-2\omega t+2\phi)$, where $A_3(y_*)$ is periodic in y_*, $A_3(y_*+4d_*) = A_3(y_*)$. The values of d_* quoted in the last paragraph are similar to the wavelength $\Lambda_3 = 2\pi/|\boldsymbol{k}_{3*}|$. For example, in the case of figure 4(*b*), $d_* = 300$ km for $U_i = 1$ cm/s and the A_3 wave of period half a year has a wavelength of 251 km. Therefore, the condition for weak nonlinearity, $4d_* \gg \Lambda_3$, is only marginally satisfied for $U_i = O(1 \text{ cm/s})$ of annual incident Rossby waves.

The steady flow of $O(\epsilon)$ is given by (2.8) but with B_{12} replaced by $\mathscr{B}_{12}(\epsilon y)$, i.e. there is an amplitude modulation in the steady flow. Thus, in case of resonance, the Eulerian steady current parallel to the wall of Mysak & Magaard (1983) in the absence of friction gets modified.

5. Energetics

In GZ the energy E and energy flux $\mathscr{S}$, expressions for which are given in the Appendix, have been calculated for $\psi^{(0)}$ given by (2.2). It is shown that the total energy inside the volume V, defined by the parallelepiped $-\pi/|k| \leqslant x \leqslant \pi/|k|$, $0 \leqslant y \leqslant y_B$, $-1 \leqslant z \leqslant 0$, remains constant in time for all $y_B > 0$. Then, it must be true that $\iint_A \mathscr{S}\cdot\hat{\boldsymbol{n}}_A\,\mathrm{d}A = 0$, where A is the area enclosing V and $\hat{\boldsymbol{n}}_A$ is the outward unit normal to A. Because $\psi^{(0)}$ is periodic in x the contributions of $\mathscr{S}_x$ (the x-component of $\mathscr{S}$) at the planes $x = -\pi/|k|, \pi/|k|$ will cancel out. At the top and bottom $\mathscr{S}_z$ vanishes; at the wall $y = 0$, $\mathscr{S}_y = 0$. Thus, $\int_{-1}^{0} \langle \mathscr{S}_y\rangle^x\,\mathrm{d}z = 0$, where the x-integral is indicated by $\langle\;\rangle^x$.

In general the enstrophy is not conserved for bounded domains (Pedlosky 1987). However, it is interesting to note that for the linear solution of the reflection problem, i.e. for $\psi^{(0)}$ given by (2.2), the total enstrophy inside V remains constant in time. Thus, although a Rossby wave packet reflected from a rigid boundary changes its total wavenumber thereby changing its enstrophy, the total enstrophy inside V is a constant in time.

Now there are three waves (see (4.3)) with their amplitudes varying slowly in y. Without going into any calculation one can deduce that $\partial_t \iiint_V E\,\mathrm{d}V = 0$ (there will be extra terms in the expression for E given by (A 4) due to the amplitude modulation, but when integrated, the time dependence drops out). The arguments above still apply for $\mathscr{S}_x$ and $\mathscr{S}_z$, since e.g. $\psi^{(0)}$ given by (4.3) is periodic in x. What about the energy flux through the plane parallel to the wall $y = y_B$ of area $-\pi/|k| \leqslant x \leqslant \pi/|k|$, $-1 \leqslant z \leqslant 0$?

To leading order in ϵ, the x-average† of $\mathscr{S}_y$ is (adapted from (A 7))

$$\begin{aligned}\langle \mathscr{S}_y\rangle^x &= \langle \mathscr{S}_{1y}\rangle^x + \langle \mathscr{S}_{2y}\rangle^x + \langle \mathscr{S}_{3y}\rangle^x \\ &= -\tfrac{1}{4}\Psi_n^2(l_1-l_2)\,\omega(A_1^2-A_2^2)+\tfrac{3}{4}\Psi_M^2\sin\alpha A_3^2. \end{aligned} \tag{5.1}$$

Note that the cross-terms $\sim A_i A_3$, $i = 1,2$ vanish when x-averaged. Therefore, the integral constraint (4.18) and

$$\int_{-1}^{0} \langle \mathscr{S}_y\rangle^x\,\mathrm{d}z = 0\ \forall y \tag{5.2}$$

are equivalent‡ statements that the leading-order total energy flux through an (x,z)-plane is zero, as required by energy conservation in the volume V.

† An x-average if the x-domain is $(-\infty,\infty)$; if integration is over a period of length $2\pi/|k|$, one multiplies the period by the x-average to get the total flux.

‡ This follows from the relation $\int_{-1}^{0}\langle\mathscr{S}\rangle^x\,\mathrm{d}z = c_g\int_{-1}^{0}\langle E\rangle^x\,\mathrm{d}z = c_g K^2A^2/4$ for a baroclinic Rossby wave.

To the next order, i.e. to $O(\epsilon)$, there will be additional terms in $\mathscr{S}$ besides those appearing in (A 6), due to the slowly varying wave amplitudes. All these terms are $(O\epsilon)$ and will be distinguished by their Y_1-derivatives. For example, the y-component of the energy flux of the ith wave is now

$$\mathscr{S}_{iy} = -A_i^2(\omega_i l_i + \tfrac{1}{2}\sin\alpha)\,\Psi_{n_i}^2 \cos^2\theta_i^{(0)} - \epsilon A_i^3(|\boldsymbol{k}_i|^2 + L^2 f_0^2 \lambda_{n_i}) k_i \Psi_{n_i}^3 \times \sin\theta_i^{(0)} \cos^2\theta_i^{(0)} - \epsilon A_i \partial_{Y_1} A_i \Psi_{n_i}^2 \omega_i \cos\theta_i^{(0)} \sin\theta_i^{(0)}, \quad i = 1,2,3. \quad (5.3)$$

When $\mathscr{S}_{iy}$ is x-averaged, the $O(\epsilon)$ terms vanish; the new terms $\sim \partial_{Y_1} A_i$ do not contribute to $\langle \mathscr{S}_{iy} \rangle^x$. Using (A 6) and taking into account the new terms, one obtains, to $O(\epsilon)$, the x-average of $\mathscr{S}_y$:

$$\begin{aligned}\langle \mathscr{S}_y \rangle^x = {} & \langle \mathscr{S}_{1y} \rangle^x + \langle \mathscr{S}_{2y} \rangle^x + \langle \mathscr{S}_{3y} \rangle^x \\ & + \epsilon \tfrac{1}{2}\omega \Psi_n^2 [A_1 \partial_{Y_1} A_2 \sin(\theta_2^{(0)} - \theta_1^{(0)}) + A_2 \partial_{Y_1} A_1 \sin(\theta_1^{(0)} - \theta_2^{(0)})] \\ & - \epsilon \tfrac{1}{4} A_1 A_3 \Psi_n \Psi_M [\hat{\boldsymbol{k}} \times \boldsymbol{k}_1 (|\boldsymbol{k}_1|^2 + L^2 f_0^2 \lambda_n + |\boldsymbol{k}_3|^2 + L^2 f_0^2 \lambda_M) A_1 \Psi_n \\ & \times \sin(\theta_1^{(0)} - \theta_2^{(0)}) + \hat{\boldsymbol{k}} \times \boldsymbol{k}_3 (|\boldsymbol{k}_1|^2 + L^2 f_0^2 \lambda_n) A_1 \Psi_n \sin(\theta_2^{(0)} - \theta_1^{(0)})] \cdot \hat{\boldsymbol{j}} \\ & - \epsilon \tfrac{1}{4} A_2 A_3 \Psi_n \Psi_M [\hat{\boldsymbol{k}} \times \boldsymbol{k}_2 (|\boldsymbol{k}_2|^2 + L^2 f_0^2 \lambda_n + |\boldsymbol{k}_3|^2 + L^2 f_0^2 \lambda_M) A_2 \Psi_n \\ & \times \sin(\theta_2^{(0)} - \theta_1^{(0)}) + \hat{\boldsymbol{k}} \times \boldsymbol{k}_3 (|\boldsymbol{k}_2|^2 + L^2 f_0^2 \lambda_n) A_2 \Psi_n \sin(\theta_1^{(0)} - \theta_2^{(0)})] \cdot \hat{\boldsymbol{j}}. \quad (5.4)\end{aligned}$$

The vector $\boldsymbol{k}_3$ has components $(2k, l_1 + l_2)$. Among the terms that vanish when $\mathscr{S}_y$ is x-averaged are those proportional to $A_i \partial_{Y_1} A_3$ and $A_3 \partial_{Y_1} A_i$, $i = 1,2$. Using that $\hat{\boldsymbol{k}} \times \boldsymbol{k}_i \cdot \hat{\boldsymbol{j}} = k_i$, equation (4.12), and after some rearrangement, the $O(\epsilon)$ terms on the right-hand side of (5.4) can be written as

$$\begin{aligned}\text{RHS} = \epsilon \Bigg\{ & \left[\tfrac{1}{2}\omega \Psi_n^2 A_1 \partial_{Y_1} A_2 + \tfrac{1}{4} A_1 A_1 A_3 \Psi_n^2 \Psi_M k \frac{\sin\alpha}{2\omega}(l_1 - l_2) \right] \sin(\theta_2^{(0)} - \theta_1^{(0)}) \\ & + \left[\tfrac{1}{2}\omega \Psi_n^2 A_2 \partial_{Y_1} A_1 - \tfrac{1}{4} A_2 A_2 A_3 \Psi_n^2 \Psi_M k \frac{\sin\alpha}{2\omega}(l_1 - l_2) \right] \sin(\theta_1^{(0)} - \theta_2^{(0)}) \Bigg\},\end{aligned}$$

from where it follows that, upon using the amplitude equations (4.13) and (4.14),

$$\int_{-1}^{0} \text{RHS}\,\mathrm{d}z = 0.$$

This in turn implies, in view of (5.1), (5.2) and (5.4), that

$$\int_{-1}^{0} \langle \mathscr{S}_y \rangle^x \,\mathrm{d}z = 0 \;(\forall y) \quad \text{to} \quad O(\epsilon). \quad (5.5)$$

The following interpretation regarding the meaning of the amplitude equations is conceivable: to $O(\epsilon)$, i.e. when first-order nonlinear interactions among the waves are taken into account, the amplitude equations assure that the total energy flux through an (x, z)-plane is a constant equal to zero, as required by energy conservation.

In summary, the integral constraint (4.18) is a statement of energy conservation to leading order in ϵ and the amplitude equations assure energy conservation up to $O(\epsilon)$.

6. Discussion and conclusions

When the third wave excited by the wave–wave interaction between an incident and a reflected Rossby wave is free, a pedestrian expansion predicts a slow linear growth of its amplitude in the offshore direction (y). This would imply an infinite average energy over the half-space, unacceptable on physical grounds. Using the method of multiple scales, it is shown that the straightforward expansion gives a correct description near the wall, but at a certain distance from it the amplitudes of the wave triad oscillate slowly in y, in a way such that the energy flux of the triad through any plane parallel to the wall vanishes, as required by energy conservation. This amplitude modulation is the envelope of a wave packet. In other words, for an incident, reflected and excited wave to form a resonantly interacting triad, it is necessary that they be a triad of wave packets. In the non-resonant case the wave amplitudes are constant in y.

Very near the coast, the energy is present mostly in the original Rossby wave pair, with the forced wave having relatively very small amplitude. With increasing distance from the coast, the forced wave becomes increasingly important whereas the reflected (incident) wave has less and less energy for a western (eastern) wall until the forced wave amplitude reaches a maximum; then the reverse happens and so on. The pulsation of energy in space is perpetual with the envelopes of the wave packets oscillating around zero except for the longest of the three waves.

The wave amplitudes cannot be functions of T_1. From the physical point of view, this means that the three waves do not exchange energy in time due to the additional constraint on the motion imposed by the boundary condition at the wall. The presence of the boundary inhibits energy exchange among the components of the wave triad, a result consistent with the statement that Rossby waves will be stabilized somewhat by the presence of boundaries (Gill 1974).

Resonant interactions (recall that this can happen only if $0 < |\sin\alpha| \leqslant \frac{1}{3}$) are the most important at first order. All others, for ϵ small, will produce an $O(\epsilon)$ field of forced waves, providing a small correction to $\psi^{(0)}$. Of all forced solutions, the resonant response would be the largest. How could such an idea be tested? The following 'thought' experiment is a possibility to do the testing. Imagine a rotating wave tank (the surface of equilibrium of such a tank would provide the β-effect) filled with homogeneous fluid, having only one effectively reflecting boundary and with a Rossby wavemaker producing continuously a Rossby wave at one of the tank's walls. To have only one reflecting boundary one could devise sponge layers at the other walls, or have a tank of large dimensions relative to the reflecting wall, etc. The frequency ω of the generated waves is determined by the frequency of the wavemaker; the wavenumber vector of the waves is determined by the wavemaker's orientation and the radiation condition. For given ω, there are at most two k values, i.e. two wave vectors, such that there is resonance (and for $0 < |\sin\alpha| \leqslant \frac{1}{3}$, where α would be the angle between the reflecting boundary and the depth contours). The wavemaker could be oriented accordingly. The theory developed here would predict that for those orientations, one would observe the maximum response.

For oceanic applications, there are some boundaries in the ocean such that resonance is possible. The following boundaries (outside the latitudinal belt between 10° S and 10° N, for quasi-geostrophic dynamics to be ensured) satisfy $0 \leqslant |\sin\alpha| \leqslant \frac{1}{3}$ with the corresponding α in parentheses: (*a*) Atlantic Ocean: N. coast of Cuba and Greater Antillas (19°), coast of Venezuela (9°); (*b*) Indian Ocean: S. coast of South Africa (173°), Great Australian Bight (167°), NW coast of Western Australia (341°); (*c*) Pacific Ocean: Aleutian Is. (166°).

Other authors have used a different approach to study the effect of boundaries on resonant interactions among Rossby waves. Plumb (1977) and Mysak (1978) took a triad of wave modes, allowing each wave mode amplitude to be slowly varying in time, which was possible since each wave mode satisfied all the boundary conditions. There was energy exchange between the wave modes. What would happen if three incident reflected Rossby wave pairs (or modes) were taken? A speculative answer follows.

First, there will be resonant interactions between the three pairs based on the fact that Plumb (1977) found resonant triads and he had the additional constraint that l was discretized. It is anticipated that the existence of resonant wave pair triads will depend on the orientation of the wall. The amplitudes of the pairs will change on a timescale of $O(\epsilon^{-1})$, with the triad of pairs exchanging energy in such a way that energy and perhaps enstrophy (of the triad) are conserved. For a non-zonal wall, the self-interaction of each wave pair or mode will generate an x-independent flow (i.e. parallel to the wall) with its phase being time independent and with amplitude proportional to the square of the pair amplitude, thus varying slowly in time. As a result, there will be no steady flow parallel to the wall.

Whether a sum of incident–reflected wave pairs would be a more suitable representation (in the reflection problem) of the real ocean than, say, one pair only, is probably so; but this is not the issue of this paper: more simple matters, i.e. the case of one incident–reflected wave pair, should be investigated and understood first.

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Appendix. Formulae for energetics

If the QGPVE is multiplied by ψ one can arrive at an energy equation of the form

$$\partial_t E + \nabla_3 \cdot \mathscr{S} = 0, \tag{A 1}$$

where $E = \frac{1}{2}\{|\nabla\psi|^2 + (\partial_z \psi)^2/S\}$ is the total energy composed of kinetic energy plus available potential energy density, ∇_3 is the three-dimensional nabla operator and

$$\mathscr{S} = -\psi\,\partial_t\left(\nabla\psi + \hat{\boldsymbol{k}}\frac{1}{S}\partial_z\psi\right) - \tfrac{1}{2}(\hat{\boldsymbol{\imath}}\cos\alpha + \hat{\boldsymbol{\jmath}}\sin\alpha)\,\psi^2 - \epsilon\psi\boldsymbol{u}\left[\nabla^2\psi + \partial_z\left(\frac{1}{S}\partial_z\psi\right)\right] \tag{A 2}$$

is the three-dimensional total energy flux vector (defined up to an arbitrary non-divergent vector), in which $\hat{\boldsymbol{k}}$ is the unit vector in the z-direction and $\boldsymbol{u} = \hat{\boldsymbol{k}} \times \nabla\psi$.

For a superposition of two arbitrary Rossby waves, whose streamfunction is

$$\begin{aligned}\psi &= A_1\,\Psi_{n_1}(z)\cos(\boldsymbol{k}_1\cdot\boldsymbol{x} - \omega_1 t + \phi_1) + A_2\,\Psi_{n_2}(z)\cos(\boldsymbol{k}_2\cdot\boldsymbol{x} - \omega_2 t + \phi_2)\\ &\equiv A_1\,\Psi_{n_1}(z)\cos\Theta_1 + A_2\,\Psi_{n_2}(z)\cos\Theta_2,\end{aligned} \tag{A 3}$$

the expression for E is given by

$$E = E_1 + E_2 + A_1 A_2\left[\boldsymbol{k}_1\cdot\boldsymbol{k}_2\,\Psi_{n_1}\Psi_{n_2}\sin\Theta_1\sin\Theta_2 + \frac{1}{S}\frac{\mathrm{d}\Psi_{n_1}}{\mathrm{d}z}\frac{\mathrm{d}\Psi_{n_2}}{\mathrm{d}z}\cos\Theta_1\cos\Theta_2\right], \tag{A 4}$$

where, for $i = 1, 2$,

$$E_i = \frac{A_i^2}{2}\left[(|\boldsymbol{k}_i|^2\sin^2\Theta_i + L^2 f_0^2\,\lambda_{n_i}\cos^2\Theta_i)\,\Psi_{n_i}^2 + \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{S}\Psi_{n_i}\frac{\mathrm{d}\Psi_{n_i}}{\mathrm{d}z}\right)\cos^2\Theta_i\right]. \tag{A 5}$$

Clearly $E \neq E_1 + E_2$. The energy of the sum is not the sum of the energies. This fact is not surprising since the energy is a quadratic functional of ψ.

On the other hand, the energy flux is

$$\begin{aligned}\mathscr{S} = {} & \mathscr{S}_1+\mathscr{S}_2-A_1 A_2(\omega_1 \boldsymbol{k}_1+\omega_2 \boldsymbol{k}_2+\hat{\boldsymbol{\imath}}_E)\Psi_{n_1}\Psi_{n_2}\cos\Theta_1\cos\Theta_2 \\ & -A_1 A_2 \hat{\boldsymbol{k}}\frac{1}{S}\left(\omega_1 \Psi_{n_2}\frac{d\Psi_{n_1}}{dz}\sin\Theta_1\cos\Theta_2+\omega_2 \Psi_{n_1}\frac{d\Psi_{n_2}}{dz}\sin\Theta_2\cos\Theta_1\right) \\ & -\epsilon\tfrac{1}{4}A_1 A_2 \Psi_{n_1}\Psi_{n_2}\Big(\hat{\boldsymbol{k}}\times\boldsymbol{k}_1\{(|\boldsymbol{k}_1|^2+L^2 f_0^2\lambda_{n_1}+|\boldsymbol{k}_2|^2+L^2 f_0^2\lambda_{n_2})A_1\Psi_{n_1} \\ & \times[\sin(2\Theta_1+\Theta_2)+\sin(2\Theta_1-\Theta_2)] \\ & +(|\boldsymbol{k}_2|^2+L^2 f_0^2\lambda_{n_2})A_2\Psi_{n_2}[2\sin\Theta_1+\sin(\Theta_1+2\Theta_2)+\sin(\Theta_1-2\Theta_2)]\} \\ & +\hat{\boldsymbol{k}}\times\boldsymbol{k}_2\{(|\boldsymbol{k}_1|^2+L^2 f_0^2\lambda_{n_1}+|\boldsymbol{k}_2|^2+L^2 f_0^2\lambda_{n_2})A_2\Psi_{n_2} \\ & \times[\sin(2\Theta_2+\Theta_1)+\sin(2\Theta_2-\Theta_1)] \\ & +(|\boldsymbol{k}_1|^2+L^2 f_0^2\lambda_{n_1})A_1\Psi_{n_1}[2\sin\Theta_2+\sin(\Theta_2+2\Theta_1)+\sin(\Theta_2-2\Theta_1)]\}\Big), \quad (A\ 6)\end{aligned}$$

where

$$\begin{aligned}\mathscr{S}_i = {} & -A_i^2\left[(\omega_i \boldsymbol{k}_i+\tfrac{1}{2}\hat{\boldsymbol{\imath}}_E)\Psi_{n_i}^2\cos^2\Theta_i+\omega_i\hat{\boldsymbol{k}}\frac{1}{S}\Psi_{n_i}\frac{d\Psi_{n_i}}{dz}\sin\Theta_i\cos\Theta_i\right] \\ & -\epsilon A_i^3(|\boldsymbol{k}_i|^2+L^2 f_0^2\lambda_{n_i})\hat{\boldsymbol{k}}\times\boldsymbol{k}_i\Psi_{n_i}^3\sin\Theta_i\cos^2\Theta_i, \quad i=1,2. \quad (A\ 7)\end{aligned}$$

Again, as in the case of the energy, the energy flux of the superposition is not the superposition of the energy fluxes: $\mathscr{S} \neq \mathscr{S}_1+\mathscr{S}_2$.

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